Journal of Combinatorial Theory, Series A **84**, 111–132 (1998)
Article No. TA972854

Inseparable Orthogonal Matrices over $\mathbb{Z}_2$

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Communicated by the Managing Editors

Received July 29, 1996

A (0, 1)-matrix A is called orthogonal over $\mathbb{Z}_2$ if both AA^T and A^TA are diagonal matrices. A matrix A is called inseparable if A contains no zero row or zero column and there do not exist permutation matrices P and Q such that

A matrix A is said to be of type 0 if $AA^T = O$ and $A^TA = O$. A square matrix A of order n is said to be of type 1 if $AA^T = I_n$. It turns out that an inseparable orthogonal matrix over $\mathbb{Z}_2$ is either of type 0 or of type 1. Let $f_0(m, n)$ (respectively, $F_0(m, n)$) denote the smallest (respectively, largest) number of 1's in an $m \times n$ inseparable orthogonal matrix of type 0 over $\mathbb{Z}_2$, and $f_1(n)$ (respectively, $F_1(n)$) denote the smallest (respectively, largest) number of 1's in an $n \times n$ inseparable orthogonal matrix of type 1 over $\mathbb{Z}_2$. The formulas for $f_0(m, n)$, $F_0(m, n)$, $f_1(n)$, and $F_1(n)$ are completely determined in this paper.

1. INTRODUCTION

In this paper we consider matrices over a given field F. A *line* of a matrix A is either a row or a column of A. Given any $m \times n$ matrix $A = (a_{ij})$, we use $R_i = R_i(A)$ to denote the ith row of A for $1 \leqslant i \leqslant m$, and $C_j = C_j(A)$ to denote the jth column of A for $1 \leqslant j \leqslant n$. We also define $\Omega(R_i) = \{C_j \mid a_{ij} \neq 0\}$ for each $1 \leqslant i \leqslant m$ and $\Omega(C_j) = \{R_i \mid a_{ij} \neq 0\}$ for each $1 \leqslant j \leqslant n$. We say a column C (respectively, a row R) *covers* a row R (respectively, a column C) if $R \in \Omega(C)$. We define $\omega(L_1, L_2) = |\Omega(L_1) \cap \Omega(L_2)|$, where L_1 and L_2 are either two columns or two rows in a matrix A. The *weight* of a line L, $\omega(L)$, is the total number of nonzero entries in L. We also define $\tau(C) = \sum_{R \in \Omega(C)} \omega(R)$ for any column C and $\tau(R) = \sum_{C \in \Omega(R)} \omega(C)$ for any row R. The *weight* of a matrix A, $\omega(A)$, is the total number of nonzero entries in A.

We use O to denote a zero matrix, I_n to denote the identity matrix of order n, and $J_{m,n}$ to denote the $m \times n$ all 1 matrix. We abbreviate $J_{n,n}$ as

0097-3165/98 $25.00

J_n. A matrix A is called *orthogonal* if both AA^T and A^TA are diagonal matrices. A matrix A is said to be of *type* 0 if $AA^T=O$ and $A^TA=O$. A matrix A of order n over F is said to be of *type* 1 if $AA^T=A^TA=cI_n$ for some nonzero constant c in F. Matrices A and B are said to be *p-equivalent* if there exist permutation matrices P and Q such that $PAQ=B$. A matrix A of order n is called *fully indecomposable* if A is not p-equivalent to a matrix of the form $\left(\begin{smallmatrix} B & C \\ O & D\end{smallmatrix}\right)$, where B is a matrix of order k, $1\leqslant k\leqslant n-1$. A matrix A is called *admissible* if A contains no zero row nor zero column. A matrix A is called *inseparable* if A is admissible and A is not p-equivalent to a matrix of the form $\left(\begin{smallmatrix} B & O \\ O & C\end{smallmatrix}\right)$, and *separable* otherwise. Clearly, a fully indecomposible matrix is necessarily inseparable. We first state the following result from [5].

LEMMA 1.1. *Let A be a fully indecomposable, orthogonal matrix over a field F. Then A is either of type* 0 *or of type* 1.

LEMMA 1.2. *Let A be an inseparable orthogonal matrix over a field F. If A is not of type* 0, *then A is fully indecomposable and of type* 1.

Proof. Let A be an $m\times n$ inseparable orthogonal matrix over F. Assume

$$AA^T=\begin{pmatrix} D_1 & O \\ O & O\end{pmatrix},\qquad A^TA=\begin{pmatrix} D_2 & O \\ O & O\end{pmatrix},\qquad \text{and}\qquad A=\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4\end{pmatrix},$$

where D_1 (respectively, D_2) is a diagonal matrix of order s (respectively, t) such that each diagonal entry of D_1 (respectively, D_2) is nonzero, and A_1 is the upper-left $s\times t$ submatrix of A. Then we have

$$AA^TA=A\begin{pmatrix} D_2 & O \\ O & O\end{pmatrix}=\begin{pmatrix} A_1D_2 & O \\ A_3D_2 & O\end{pmatrix}$$

and

$$AA^TA=\begin{pmatrix} D_1 & O \\ O & O\end{pmatrix}A=\begin{pmatrix} D_1A_1 & D_1A_2 \\ O & O\end{pmatrix},$$

which implies that $A_2=O$ and $A_3=O$. Therefore,

$$A=\begin{pmatrix} A_1 & O \\ O & A_4\end{pmatrix}.$$

Since A is inseparable and A is not of type 0, it follows that $s=m$ and $t=n$. Therefore, we have $AA^T=D_1$ and $A^TA=D_2$. Since the m rows of A are linearly independent, it follows that $m\leqslant n$. Similarly, we have $n\leqslant m$ since the n columns of A are also linearly independent. Therefore, we have $m=n$ and A is an invertible matrix of order n.

We now claim that A is fully indecomposable. Suppose not, then there exists a submatrix B of order k, $1 \leqslant k \leqslant n-1$, such that A is p-equivalent to a matrix of the form $\left(\begin{smallmatrix} B & C \\ O & D \end{smallmatrix}\right)$. Without loss of generality, we assume $A = \left(\begin{smallmatrix} B & C \\ O & D \end{smallmatrix}\right)$. Then we see that $C \neq O$ and D is invertible. Since

$$AA^T = \begin{pmatrix} BB^T + CC^T & CD^T \\ DC^T & DD^T \end{pmatrix}$$

is a diagonal matrix, it follows that $DC^T = O$. This would imply that $C = O$, which is impossible. Therefore, A is fully indecomposable, and the result follows immediately from Lemma 1.1. ∎

Two vectors $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ are said to be *combinatorially orthogonal* if $|\{i : x_i y_i \neq 0\}| \neq 1$. A matrix A is *combinatorially orthogonal* if each pair of rows of A is combinatorially orthogonal and each pair of columns of A is combinatorially orthogonal. Clearly, if A is an orthogonal matrix over F, then A is also combinatorially orthogonal. Furthermore, for any combinatorially orthogonal matrix A over F, the matrix obtained from A by replacing each nonzero entry in A by a 1 is combinatorially orthogonal. We will need the following result of Reid and Thomassen [4]. The reader is referred to [3] for a simple proof of Lemma 1.3 and further results concerning inseparable and combinatorially orthogonal (0, 1)-matrices.

LEMMA 1.3. *Let A be an $m \times n$ inseparable and combinatorially orthogonal* (0, 1)*-matrix. Then* $\omega(A) \geqslant 2(m+n) - 4$.

Let $f(F; m, n)$ denote the smallest number of nonzero entries in an $m \times n$ inseparable orthogonal matrix over F. Then Lemma 1.3 implies that $f(F; m, n) \geqslant 2(m+n) - 4$ for any field F, if it exists. Trivially, we have $f(F; 2, 2) = 4$ for any field F. It was proved in [1] that $f(\mathbb{R}; n, n) = 4(n-1)$, where $\mathbb{R}$ is the field of real numbers. It can be shown that $f(\mathbb{Z}_2; 3, 3)$ does not exist and $f(F; 3, 3) \geqslant 8$ if $F \neq \mathbb{Z}_2$, with equality if and only if there exists two nonzero elements a and b in F such that $b^2 - a^2 = 1$, in which case,

$$A = \begin{pmatrix} a & 1 & a^{-1}b \\ 1 & a^{-1} & -b \\ a^{-1}b & -b & 0 \end{pmatrix}$$

is an orthogonal matrix over F.

Orthogonal $(0, 1, -1)$-matrices over $\mathbb{R}$ have been referred to as weighing matrices in the literature. Clearly, every weighing matrix can be viewed as an orthogonal matrix over any field F. Weighing matrices have been studied extensively (e.g., see [2]). It turns out that the problem of evaluating $f(F; m, n)$ over a finite field $F = GF(q)$ is very difficult. For the remainder of this paper, we will consider the case $F = \mathbb{Z}_2$ only. Note that if a matrix A of order n over $\mathbb{Z}_2$ is of type 1, then we have $AA^T = A^TA = I_n$. We also note that if matrices A and B are p-equivalent, then $\omega(A) = \omega(B)$, and A is inseparable (resp., orthogonal) if and only if B is inseparable (resp., orthogonal). Therefore, we will not distinguish p-equivalent matrices in this paper. In addition, for every statement in terms of the columns of a matrix A, there is a corresponding statement in terms of the rows of A. So we will only state our results in terms of columns in most cases.

Let $f_0(m, n)$ (respectively, $F_0(m, n)$) denote the smallest (respectively, largest) number of 1's in an $m \times n$ inseparable orthogonal matrix of type 0 over $\mathbb{Z}_2$, and $f_1(n)$ (resp. $F_1(n)$) denote the smallest (resp., largest) number of 1's in an $n \times n$ inseparable orthogonal matrix of type 1 over $\mathbb{Z}_2$. Clearly we have $f_0(m, n) = f_0(n, m)$ and $F_0(m, n) = F_0(n, m)$. So we will assume m is odd if at least one of m and n is odd. The formulas for $F_1(n)$ (Theorem 3.2), $F_0(m, n)$ (Theorem 3.4), $f_1(n)$ (Theorem 4.1), and $f_0(m, n)$ (Theorem 4.2) are completely determined in this paper.

2. SOME PRELIMINARY RESULTS

We first note that the two matrices in Fig. 1 are both inseparable and of type 0. They will be used in constructing larger matrices in this paper.

LEMMA 2.1. *For any $m \times n$ orthogonal matrix A, we have*

$$\omega(A) \equiv \begin{cases} 0 \pmod 4 & \text{if } A \text{ is of type } 0 \\ n \pmod 4 & \text{if } A \text{ is a square matrix of type } 1. \end{cases}$$

Proof. Let $k_i = \omega(C_i)$ for $1 \leqslant i \leqslant n$, and let λ_{ij} be the total number of columns covering both row i and row j for $1 \leqslant i \neq j \leqslant m$. Then by counting

$$A_{7,6} = \begin{pmatrix} 1&1&1&1&1&1 \\ 1&1&1&1&0&0 \\ 1&1&0&0&1&1 \\ 1&1&0&0&0&0 \\ 0&0&1&1&1&1 \\ 0&0&1&1&0&0 \\ 0&0&0&0&1&1 \end{pmatrix} \qquad A_7 = \begin{pmatrix} 1&1&1&1&0&0&0 \\ 1&1&0&0&1&1&0 \\ 1&0&1&0&1&0&1 \\ 1&0&0&1&0&1&1 \\ 0&1&1&0&0&1&1 \\ 0&1&0&1&1&0&1 \\ 0&0&1&1&1&1&0 \end{pmatrix}$$

(a) (b)

FIGURE 1

the number s of submatrices of the form $J_{2,1}$ (i.e., the number of columns in A covering an unordered pair of rows) in two different ways, it follows that

$$s = \sum_{i=1}^{n} \binom{k_i}{2} = \sum_{1 \leqslant i < j \leqslant m} \lambda_{ij}.$$

Since A is orthogonal, it follows that s is even. Therefore, we have

$$\sum_{i=1}^{n} (k_i^2 - k_i) = 2s \equiv 0 \pmod 4,$$

which implies that

$$\omega(A) = \sum_{i=1}^{n} k_i \equiv \sum_{i=1}^{n} k_i^2 \pmod 4.$$

Note that either every k_i is even or every k_i is odd. Therefore, we have

$$\omega(A) \equiv \begin{cases} 0 \pmod 4 & \text{if } A \text{ is of type } 0 \\ n \pmod 4 & \text{if } A \text{ is a square matrix of type } 1. \end{cases}$$ ▮

LEMMA 2.2. *For any $m \times n$ orthogonal matrix A and any column C in A, we have*

$$\tau(C) \equiv \begin{cases} 0 \pmod 4 & \textit{if } A \textit{ is of type } 0 \\ 1 \pmod 4 & \textit{if } A \textit{ is a square matrix of type } 1. \end{cases}$$

Proof. Let C be an arbitrary column in A. Define $p_i = \omega(C, C_i)$ for $1 \leqslant i \leqslant n$. Without loss of generality, we assume $C = C_1$ and $\Omega(C_1) = \{R_i \mid 1 \leqslant i \leqslant p_1\}$. Observe that p_i is even for each i, $2 \leqslant i \leqslant n$, since A is orthogonal. We now define λ_{ij} to be the total number of columns in A covering both row i and row j for $1 \leqslant i \neq j \leqslant p_1$. Then by counting the number s of columns in A covering an unordered pair of rows in $\Omega(C)$ in two different ways, it follows that

$$s = \sum_{i=1}^{n} \binom{p_i}{2} = \sum_{1 \leqslant i < j \leqslant p_1} \lambda_{ij}.$$

Since A is orthogonal, it follows that s is even. Therefore, we have

$$\sum_{i=1}^{n} (p_i^2 - p_i) = 2s \equiv 0 \pmod 4,$$

which implies that

$$\tau(C) = \sum_{i=1}^{n} p_i \equiv \sum_{i=1}^{n} p_i^2 \equiv p_1^2 \pmod 4.$$

Therefore, we have

$$\tau(C) \equiv \begin{cases} 0 \pmod 4 & \text{if } A \text{ is of type } 0 \\ 1 \pmod 4 & \text{if } A \text{ is a square matrix of type } 1. \end{cases} \quad \blacksquare$$

The following result can be easily verified.

LEMMA 2.3. *Let $A \neq I_1$ be an $m \times n$ admissible orthogonal matrix such that m is odd. If A is either inseparable or of type* 0, *then we have $m \geqslant 7$ and $n \geqslant 6$.*

For the remainder of this paper, we will assume that $m \geqslant 7$ and $n \geqslant 6$, if m is odd.

LEMMA 2.4. *Let A be a $7 \times n$ admissible orthogonal matrix of type* 0, *then the weight of every column of A is exactly* 4.

Proof. It is easy to see that A does not contain a column of weight 6 since A contains no zero row. Suppose A contains a column of weight 2, and let t be the maximum number of identical columns of weight 2. Then we have

$$A = \begin{pmatrix} J_{2,t} & B_1 \\ 0 & B_2 \end{pmatrix},$$

where B_2 is a $5 \times (n-t)$ orthogonal matrix of type 0. So B_2 contains a 0 column by Lemma 2.3. But this is impossible since A is admissible and t is maximal. Therefore, the weight of every column of A is exactly 4. $\blacksquare$

The following result is proved in [6].

LEMMA 2.5. *If there exists an inseparable orthogonal matrix A of odd order n such that the weight of every line in A is exactly* 4, *then $n = 7$ and A is p-equivalent to A_7.*

Let $A = (a_{ij})$ be a square matrix of order m such that $a_{m1} = 1$, and $B = (b_{ij})$ be a square matrix of order n such that $b_{1n} = 1$. We define a square matrix $C = (c_{ij}) = A \oplus B$ of order $m + n - 1$ as follows:

$$c_{ij}=\begin{cases} a_{i(j-n+1)} & \text{if} \quad i\leqslant m \quad \text{and} \quad j\geqslant n \\ b_{(i-m+1)\,j} & \text{if} \quad i\geqslant m \quad \text{and} \quad j\leqslant n \\ 0 & \text{if} \quad i>m \quad \text{and} \quad j>n \\ 1 & \text{if} \quad i<m;\ j<n; \quad \text{and} \quad a_{i1}=b_{1j}=1 \\ 0 & \text{if} \quad i<m;\ j<n; \quad \text{and either} \quad a_{i1}=0 \quad \text{or} \quad b_{1j}=0. \end{cases}$$

or equivalently, suppose

$$A=\begin{pmatrix} A_1 & A_2 \\ 1 & A_3 \end{pmatrix}, \qquad B=\begin{pmatrix} B_1 & 1 \\ B_2 & B_3 \end{pmatrix}, \qquad \text{and}$$

$Q=(q_{ij})$ is an $(m-1)\times(n-1)$ matrix such that $q_{ij}=1$ if and only if $a_{i1}=b_{1j}=1$. Then we define

$$A\oplus B=\begin{pmatrix} Q & A_1 & A_2 \\ B_1 & 1 & A_3 \\ B_2 & B_3 & O \end{pmatrix}.$$

LEMMA 2.6. *Let $A=(a_{ij})$ be an inseparable orthogonal matrix of order m and type 1 such that $a_{m1}=1$, and $B=(b_{ij})$ be an inseparable orthogonal matrix of order n and type 1 such that $b_{1n}=1$. Then $A\oplus B$ is an inseparable orthogonal matrix of order $m+n-1$ and type 1. Moreover, we have $\omega(A\oplus B)=\omega(A)+\omega(B)+ab-a-b$, where a is the weight of the first column of A and b is the weight of the first row of B.*

Proof. It can be easily seen that $A\oplus B$ is an inseparable orthogonal matrix of order $m+n-1$ and type 1. We also have

$$\begin{aligned}\omega(A\oplus B)&=\omega(A)+\omega(B)-1+(a-1)(b-1)\\ &=\omega(A)+\omega(B)+ab-a-b. \quad \blacksquare\end{aligned}$$

3. UPPER BOUNDS

3.1. *Matrices of Type* 1

LEMMA 3.1. *Let A be a matrix of order n and type 1, where $n\geqslant 7$ is odd, such that the weight of every line of A is at least 3. Then we have $\omega(A)\leqslant n^2-5n+13$.*

Proof. We use induction on n. Clearly A contains no two identical lines, and the weight of each line of A is at most $n-2$. We may assume that $n\geqslant q$, since the case $n=7$ can be easily verified. Now let $r=r(A)$

(respectively, $c = c(A)$) denote the number of rows (respectively, columns) in A of weight $n-2$. We consider three cases.

Case 1. $r(A) = c(A) = 0$.

If the weight of every column is at most $n-6$, then we have $\omega(A) \leqslant n(n-6) \leqslant n^2 - 5n + 13$. So we assume A contains a column C_1 with weight $n-4$. Without loss of generality, we assume $\Omega(C_1) = \{R_5, R_6, ..., R_n\}$. Then each remaining column of A covers at most three of the first 4 rows in A. So the number of 1's among the first 4 rows of A is at most $3(n-1)$. Note that the weight of each row of A is at most $n-4$. Therefore, we have

$$\omega(A) \leqslant 3(n-1) + (n-4)(n-4) = n^2 - 5n + 13.$$

Case 2. $r(A) \leqslant 3$, and $c(A) \leqslant 3$.

Without loss of generality, we assume $\Omega(C_1) = \{R_3, R_4, ..., R_n\}$. Then each remaining column of A covers exactly one of the first 2 rows in A. So the number of 1's among the first 2 rows of A is exactly $n-1$. Note that the number of 1's among the remaining $n-2$ rows of A is at most $(n-4)(n-2)+6$ since $r(A) \leqslant 3$. Therefore, we have $\omega(A) \leqslant n-1+(n-4)(n-2)+6 = n^2 - 5n + 13$.

Case 3. $r(A) \geqslant 4$, or $c(A) \geqslant 4$.

Clearly, we can assume that $c(A) \geqslant 4$, and A has the following format:

$$A = \left(\begin{array}{cc|cc} 1 & 0 & J_{2,t} & O \\ 0 & 1 & & \\ 0 & 0 & O & J_{1,(n-t-2)} \\ \hline 1 & 1 & & \\ \vdots & \vdots & & X \\ 1 & 1 & & \end{array}\right).$$

Since $c(A) \geqslant 4$, it follows that if C_i is a column with weight $n-2$, then $i \leqslant t+2$. Without loss of generality, we assume that each of the first 4 columns of A has weight $n-2$. Let B be the submatrix of order $n-2$ obtained from A by deleting the first two columns and first two rows from A. Then B is of type 1. If B contains a line of weight 1, then we can easily see that the columns C_3 and C_4 (of weight $n-2$ each) must be identical, which is impossible. Note that the third row of A has weight at least 3, which implies that $t \leqslant n-5$. Therefore, by induction we have

$$\begin{aligned}\omega(A) &= \omega(B) + 2(n-2) + 2t \\ &\leqslant (n-2)^2 - 5(n-2) + 13 + 2(n-2) + 2(n-5) \\ &= n^2 - 5n + 13. \quad \blacksquare\end{aligned}$$

THEOREM 3.2. *We have*

$$F_1(n) = \begin{cases} n^2 - n & \text{if } n \geqslant 4 \text{ is even} \\ n^2 - 5n + 13 & \text{if } n \geqslant 7 \text{ is odd.} \end{cases}$$

Proof. If A is an admissible matrix of order n and type 1, where n is even, then we have $n \geqslant 4$ and the weight of each line of A is at most $n-1$. So we have $F_1(n) \leqslant n^2 - n$ if $n \geqslant 4$ is even. It is easy to see that $J_n - I_n$ is an inseparable orthogonal matrix of type 1 when $n \geqslant 4$ is even. It then follows by Lemma 2.6 that $(J_4 - I_4) \oplus (J_{n-3} - I_{n-3})$ is an inseparable orthogonal matrix of type 1 when $n \geqslant 7$ is odd. ∎

3.2. *Matrices of Type* 0

The *complement* of a matrix A is obtained from A by replacing every 0 by a 1 and every 1 by a 0.

LEMMA 3.3. *Let B be an $m \times n$ matrix such that $BB^T = J_m$ and $B^T B = J_n$, where m and n are both odd. If the number of* 0'*s in every line of B is at least* 2, *then we have $m \geqslant 7$, $n \geqslant 7$, and $\omega(B) \geqslant 3(m+n-7)$.*

Proof. We use contradiction. Suppose there exists an $n \times n$ matrix B such that $\omega(B) < 3(m+n-7)$, $BB^T = J_m$ and $B^T B = J_n$, where m and n are both odd, and the number of 0's in every line of B is at least 2. We further assume that $m+n$ is minimal among such matrices. Let A be the complement of B. Then A is an admissible matrix of type 0. So $m \geqslant 7$ and $n \geqslant 7$ by Lemma 2.3. Clearly the weight of each line of B is odd and at least 3. Let $t_r = t_r(B)$ (respectively, $t_c = t_c(B)$) denote the number of rows (respectively, columns) of weight 3 in B.

CLAIM 1. *B contains at most two rows of weight at least $n-4$ each, and B contains at most two columns of weight at least $m-4$ each.*

Proof. If B contains at least three rows of weight at least $n-4$ each, then we would have

$$\omega(B) \geqslant 3(n-4) + (m-3)\,3 = 3(m+n-7).$$

which is impossible.

CLAIM 2. *If B contains two identical columns of weight* 3 *each, then $t_c \leqslant 4$.*

Proof. Let B_1 be the submatrix of B obtained by deleting two identical columns of weight 3. Then B_1 is an $m \times (n-2)$ matrix such that $B_1 B_1^T = J_m$ and $B_1^T B_1 = J_{n-2}$. If the number of 0's in every line of B_1 is at least 2, then we would have $\omega(B)=\omega(B_1)+6\geqslant 3(m+n-2-7)+6=3(m+n-7)$, which is impossible. So there exists a row in B_1 of weight $n-2$, which means that B contains a row of weight $n-2$, and B contains no three identical columns of weight three.

Suppose B contains another pair of identical columns of weight three. Then B has the following format (by interchanging some lines in B):

$$B=\left(\begin{array}{cccc|c} 0 & 0 & 1 & 1 & \multirow{2}{*}{$J_{2,(n-4)}$} \\ 1 & 1 & 0 & 0 & \\ \hline 1 & 1 & 1 & 1 & \\ 1 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 1 & B_2 \\ \cline{1-4} \multicolumn{4}{c|}{O} & \end{array}\right),$$

where B_2 is an $(m-2)\times(n-4)$ matrix such that $B_2B_2^T=J_{m-2}$ and $B_2^TB_2=J_{n-4}$. If $t_c\geqslant 5$, then B_2 contains a column of weight 1, which implies that the corresponding row of B_2 has weight at least $n-4$. Therefore, B contains at least three rows of weight at least $n-4$ each, which contradicts Claim 1. Therefore, we have $t_c\leqslant 4$ if B contains two different pairs of identical columns of weight three.

Now assume B contains a unique pair of identical columns of weight three. Suppose $t_c\geqslant 5$, then B must have the following format:

$$B=\left(\begin{array}{ccccc|c} 0 & 0 & 1 & 1 & 1 & 1\cdots 1 \\ \hline \multicolumn{5}{c|}{\begin{matrix} J_{3,2} & I_3 \\ O & I_3 \end{matrix}} & X \\ \hline \multicolumn{5}{c|}{O} & Y \end{array}\right).$$

This implies that each column of B covers at least three of the first 7 rows of B. So the total number of 1's in the first 7 rows of B is at least $3n$. Therefore,

$$\omega(B)\geqslant 3n+3(m-7)=3(m+n-7),$$

which is impossible. This proves Claim 2.

Claim 3. *If $t_c\geqslant 5$, then B contains no two identical columns of weight* 3, *and there exists a row in B which covers every column of weight three in B.*

Proof. Assume $t_c\geqslant 5$. Then Claim 2 implies that B contains no two identical columns of weight 3. If B contains 4 columns of weight 3, each of which covers a given row R, then every other column of B also covers row R by Claim 2, and we are done. So we assume that every row in B covers at most 3 columns of weight 3 in B. Then we can easily see that B has the following format:

$$B = \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & \\ 1 & 0 & 0 & 1 & 1 & \\ 1 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 1 & 0 & X \\ 0 & 1 & 0 & 0 & 1 & \\ 0 & 0 & 1 & 1 & 0 & \\ 0 & 0 & 1 & 0 & 1 & \\ \hline & & 0 & & & Y \end{array}\right).$$

Therefore, each column of B covers at least three of the first 7 rows of B. So the total number of 1's in the first 7 rows of B is at least $3n$. Therefore,

$$\omega(B) \geqslant 3n + 3(m-7) = 3(m+n-7),$$

which is impossible. This proves Claim 3.

Let $r = r(B)$ (respectively, $c = c(B)$) denote the minimum weight of a row (respectively, column) in B. Then it follows by Claim 1 that $r \leqslant n-6$ and $c \leqslant m-6$.

Claim 4. *We have*

$$\omega(B) \geqslant rm + n - rc + c - 1 \qquad \textit{and} \qquad \omega(B) \geqslant cn + m - rc + r - 1. \quad (1)$$

Proof. Without loss of generality, we assume that the weight of the first column C_1 of B is minimal, and C_1 consists of the first c rows of B. Then each of the remaining columns of B covers at least one of the first c rows of B. So the total number of 1's in the first c rows of B is at least $c + (n-1)$. Note that the weight of each row in B is at least r. Therefore,

$$\omega(B) \geqslant (c+n-1) + r(m-c) = rm + n - rc + c - 1.$$

By symmetry, we have $\omega(B) \geqslant cn + m - rc + r - 1$.

Claim 5. *We have*

$$\omega(B) \geqslant 5m + n - 4c - 9 \qquad \textit{and} \qquad \omega(B) \geqslant 5n + m - 4r - 9. \qquad (2)$$

Proof. We will prove the first inequality only. Without loss of generality, we assume that the weight of the firs column C_1 of B is minimal, and $\Omega(C_1)$ consists of the first c rows of B. Then each of the remaining columns of B covers at least one of the first c rows of B. So the total number of 1's in the first c rows of B is at least $c+(n-1)$. Let $\mathfrak{R}$ be the set of remaining $m-c$ rows of B. If there are at most 4 rows of weight 3 in $\mathfrak{R}$, then we have

$$\omega(B) \geqslant n+c-1+5(m-c)-8=5m+n-4c-9.$$

So we assume that there are at least 5 rows of weight 3 in $\mathfrak{R}$. Let X_1 be the set of rows of weight 3 in $\mathfrak{R}$. Then we may assume that each row of weight 3 covers the last column C_n of B by Claim 3. Let X_2 be the set of rows of weight at least 5 in $\mathfrak{R}$ covering the last column of B, and $X_3 = \mathfrak{R} \backslash (X_1 \cup X_2)$. Also let $x_i = |X_i|$ for $1 \leqslant i \leqslant 3$. Note that $x_1 \geqslant 5$, $x_1+x_2+x_3=m-c$, and the weight of each row in X_3 is at least x_1. Therefore, we have

$$\begin{aligned}\omega(B) &\geqslant n+c-1+3x_1+5x_2+x_1x_3\\ &=n+c-1+5(m-c)+(x_1-5)\,x_3-2x_1.\end{aligned}$$

If $x_3 \geqslant 2$, then we have

$$\omega(B) \geqslant n+c-1+5(m-c)-10=5m+n-4c-11.$$

Since $\omega(B) \equiv mn \pmod 4$ by Lemma 2.1, it follows that $\omega(B) \geqslant 5m+n-4c-9$ if $x_3 \geqslant 2$. Now assume $x_3 \leqslant 1$. We also assume $R_n \notin \Omega(C_n)$ if $x_3=1$. Then we have $x_2 \geqslant m-c-x_1-1$. Recall that $\Omega(C_1)=\{R_1, R_2, ..., R_c\}$ is the set of the first c rows of B. Now let $Y=\Omega(C_n) \cap \Omega(C_1) = \{R_1, R_2, ..., R_y\}$, where $y=|Y|$, $Z=\Omega(C_1) \backslash Y$, and $z=|Z|$. Then we have $y=c-z \leqslant c-2$ since the number of 0's in the last column of B is at least two and z is even. Given any column C in B, we have $\omega(C, C_1)$ and $\omega(C, C_n)$ are both odd, which implies that $|\Omega(C) \cap Y|$ is odd if $x_3=0$ and $|\Omega(C) \cap (Y \cup R_n)|$ is odd if $x_3=1$. So the total number of 1's among the first y rows (and the last row R_n if $x_3=1$) in B is at least $2y+(n-2)$. Note that the weight of each row in Z is at least (x_1+1). Therefore, we have

$$\begin{aligned}\omega(B) &\geqslant 2y+(n-2)+(x_1+1)\,z+3x_1+5x_2\\ &\geqslant 2y+(n-2)+(x_1+1)(c-y)+3x_1+5(m-c-x_1-1)\\ &=5m+n-4c+y-7+(c-2-y)\,x_1\\ &\geqslant 5m+n-4c+y-7\\ &>5m+n-4c-9.\end{aligned}$$

Similarly, we can prove that $\omega(B) \geqslant 5n+m-4r-9$. This proves Claim 5.

We now consider 3 cases.

Case 1. $r \geqslant 5$ and $c \geqslant 5$.
Since $r \leqslant n-6$ and $c \leqslant m-6$, it follows by Claim 4 that

$$\begin{aligned} 2\omega(B) &\geqslant (r+1)\, m + (c+1)\, n - 2rc + r + c - 2 \\ &= 6(m+n) - 42 + (r-5)\, m + (c-5)\, n - 2rc + r + c + 40 \\ &= 6(m+n) - 42 + (r-5)(m-c-4) + (c-5)(n-r-4) \\ &\geqslant 6(m+n) - 42. \end{aligned}$$

Therefore, we have $\omega(B) \geqslant 3(m+n-7)$, which is a contradiction.

Case 2. $r = 3$ and $c \geqslant 5$.
It follows by Claims 4 and 5 that

$$\begin{aligned} 2\omega(B) &\geqslant (cn + m - 3c + 2) + (5m + n - 4c - 9) \\ &= 6m + (c+1)\, n - 7c - 7 \\ &= 6m + 6n - 42 + (c-5)(n-7) \\ &\geqslant 6(m+n-7). \end{aligned}$$

Therefore, we have $\omega(B) \geqslant 3(m+n-7)$, which is a contradiction.

Case 3. $r = c = 3$.
it follows by Claim 5 that

$$2\omega(B) \geqslant (5m + n - 21) + (5n + m - 21) = 6(m+n-7).$$

Therefore, we have $\omega(B) \geqslant 3(m+n-7)$, which is a contradiction. ∎

THEOREM 3.4. *We have*

$$F_0(m, n) = \begin{cases} mn & \text{if } m \text{ and } n \text{ are both even} \\ (m-3)\, n & \text{if } m \geqslant 7 \text{ is odd and } n \geqslant 6 \text{ is even} \\ mn - 3m - 3n + 21 & \text{if } m \geqslant 7 \text{ and } n \geqslant 7 \text{ are both odd.} \end{cases}$$

Proof. Clearly we have $F_0(m, n) = mn$ if m and n are both even. Let A be an $m \times n$ admissible orthogonal matrix of type 0, where m is odd. Then the weight of each column of A is at most $m-3$. So $\omega(A) \leqslant (m-3)\, n$. This implies that $F_0(m,n) \leqslant (m-3)\,n$ if m is odd. So it suffices to construct the upper bound by Lemma 3.3. We define

$$A = \left(\begin{array}{c|c} A_{7,6} & \begin{matrix} J_{4,(n-6)} \\ O \end{matrix} \\ \hline J_{(m-7),6} & J_{(m-7),(n-6)} \end{array}\right), \qquad \text{where}$$

$A_{7,6}$ is as in Fig. 1a. Then A is an $m \times n$ inseparable orthogonal matrix of type 0 with $\omega(A) = (m-3)\,n$, if $m \geqslant 7$ is odd and $n \geqslant 6$ is even.

We now define

$$B = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{A_7} & \begin{matrix} J_{4,(n-7)} \\ O \end{matrix} \\ \hline J_{(m-7),4} & O & J_{(m-7),(n-7)} \end{array}\right), \qquad \text{where}$$

A_7 is as in Fig. 1b. Then B is an $m \times n$ inseparable orthogonal matrix of type 0 with $\omega(A) = mn - 3m - 3n + 21$, if $m \geqslant 7$ and $n \geqslant 7$ are both odd. ∎

4. LOWER BOUNDS

4.1. *Matrices of Type* 1

THEOREM 4.1. *We have $f_1(n) = 5n - 8$ if $n \geqslant 4$ and $n \neq 5$.*

Proof. It is easy to see that $\begin{pmatrix} I_{n-2} & J_{(n-2),2} \\ J_{2,(n-2)} & I_2 \end{pmatrix}$ is an inseparable orthogonal matrix of type 1 when $n \geqslant 4$ is even, and $(J_4 - I_4) \oplus \begin{pmatrix} I_{n-5} & J_{n-5,2} \\ J_{2,n-5} & I_2 \end{pmatrix}$ is an inseparable orthogonal matrix of type 1 when $n \geqslant 7$ is odd.

We now prove the lower bound. It is easy to verify the result for $n \leqslant 5$. So we assume $n \geqslant 6$. Suppose $n \geqslant 6$ is the smallest integer such that $f_1(n) < 5n - 8$, and let A be an inseparable orthogonal matrix of order n and type 1 such that $\omega(A) = f_1(n)$.

CLAIM 1. *A does not contain three columns C_1, C_2, and C_3 of weight 3 each such that $\omega(C_1, C_2) = \omega(C_1, C_3) = \omega(C_2, C_3) = 2$. Moreover, if A contains two columns C_1 and C_2 of weight 3 each such that $\omega(C_1, C_2) = 2$, then no column (other than C_1 and C_2) in A covers both rows in $\Omega(C_1) \cap \Omega(C_2)$.*

Proof. First assume that A contains two columns C_1 and C_2 of weight 3 each such that $\omega(C_1, C_2) = 2$, $\Omega(C_1) = \{R_1, R_3, R_4\}$, and $\Omega(C_2) = \{R_2, R_3, R_4\}$. Suppose there exists another column C covering both R_3 and R_4. Then we can assume that A has the following format:

$$A = \left(\begin{array}{cc|cc} 1 & 0 & 1\cdots 1 & 0\cdots 0 \\ 0 & 1 & 1\cdots 1 & 0\cdots 0 \\ \hline 1 & 1 & & \\ 1 & 1 & & \\ 0 & 0 & & B \\ \vdots & \vdots & & \\ 0 & 0 & & \end{array}\right).$$

Then it can be seen that B is an inseparable orthogonal matrix of order $n-2$ and type 1. Therefore, we have

$$\omega(A) \geqslant \omega(B) + 10 \geqslant 5(n-2) - 8 + 10 = 5n - 8,$$

which is impossible. Therefore, C_1 and C_2 are the only columns in A covering both R_3 and R_4.

Now suppose C_3 is another column in A with weight 3 such that $\omega(C_1, C_2) = \omega(C_1, C_3) = \omega(C_2, C_3) = 2$. Then A has the following format:

$$A = \left(\begin{array}{ccc|cc} 1 & 1 & 1 & & \\ 1 & 1 & 0 & O & O \\ 1 & 0 & 1 & J_{3,t} & O \\ 0 & 1 & 1 & & \\ \hline & O & & X & \end{array}\right),$$

where $t \geqslant 3$ is odd. Let B be the submatrix obtained by deleting the first 3 columns and the first 3 rows from A. Then it can be seen that B is an inseparable orthogonal matrix of order $n-3$ and type 1. Therefore, we have

$$\omega(A) = \omega(B) + 2t + 9 \geqslant 5(n-3) - 8 + 15 = 5n - 8,$$

which is impossible. Therefore, A does not contain three columns C_1, C_2, and C_3 with weight 3 each such that $\omega(C_1, C_2) = \omega(C_1, C_3) = \omega(C_2, C_3) = 2$.

Claim 2. *Assume A contains two columns C_1 and C_2 of weight 3 each such that $\omega(C_1, C_2) = 2$. If $\Omega(C_1) = \{R_1, R_3, R_4\}$ and $\Omega(C_2) = \{R_2, R_3, R_4\}$, then we have $\omega(R_1) = \omega(R_2) \geqslant 7$, $\omega(R_3) \geqslant 5$, and $\omega(R_4) \geqslant 5$.*

Proof. Assume that A contains two columns C_1 and C_2 of weight 3 each such that $\omega(C_1, C_2) = 2$, $\Omega(C_1) = \{R_1, R_3, R_4\}$, and $\Omega(C_2) = \{R_2, R_3, R_4\}$. Then we have

$$A=\left(\begin{array}{cc|cc}
1 & 0 & 1\cdots 1 & 0\cdots 0\\
0 & 1 & 1\cdots 1 & 0\cdots 0\\
\hline
1 & 1 & & \\
1 & 1 & & \\
0 & 0 & \multicolumn{2}{c}{B} \\
\vdots & \vdots & & \\
0 & 0 & &
\end{array}\right),$$

where B is separable according to the proof of Claim 1. Therefore, A has the following format:

$$A=\left(\begin{array}{cc|cc|cc}
1 & 0 & J_{2,\alpha} & O & J_{2,\beta} & O\\
0 & 1 & & & & \\
\hline
1 & 1 & & & & \\
0 & 0 & J_{1,\alpha} & O & \multicolumn{2}{c}{O} \\
\vdots & \vdots & \multicolumn{2}{c|}{X} & & \\
0 & 0 & & & & \\
\hline
1 & 1 & & & & \\
0 & 0 & \multicolumn{2}{c|}{O} & J_{1,\beta} & O \\
\vdots & \vdots & & & \multicolumn{2}{c}{Y} \\
0 & 0 & & & &
\end{array}\right),$$

where both α and β are odd. If $\alpha=1$, then it can be seen that the weight of the third column of A is 3, which is impossible by Claim 1. So we have $\alpha\geqslant 3$. Similarly, we have $\beta\geqslant 3$. This implies that $\omega(R_1)=\omega(R_2)=\alpha+\beta+1\geqslant 7$, $\omega(R_3)=\alpha+2\geqslant 5$, and $\omega(R_4)=\beta+2\geqslant 5$.

CLAIM 3. *If A contains a column C_1 of weight 3 and a row R_1 of weight 3 such that $\Omega(C_1)=\{R_1, R_2, R_3\}$ and $\Omega(R_1)=\{C_1, C_2, C_3\}$, then we have $\omega(R_2)=\omega(R_3)\geqslant 5$, $\omega(C_2)=\omega(C_3)\geqslant 5$, and $\omega(C_2)+\omega(R_2)\geqslant 12$.*

Proof. Assume that A contains a column C_1 of weight 3 and a row R_1 of weight 3 such that $\Omega(C_1)=\{R_1, R_2, R_3\}$ and $\Omega(R_1)=\{C_1, C_2, C_3\}$. Then A has the following format:

$$A = \left(\begin{array}{ccc|cc}
1 & 1 & 1 & 0\cdots 0 & 0\cdots 0 \\
1 & 1 & 0 & 1\cdots 1 & 0\cdots 0 \\
1 & 0 & 1 & 1\cdots 1 & 0\cdots 0 \\
\hline
0 & 1 & 1 & & \\
\vdots & \vdots & & & \\
0 & 1 & 1 & & B \\
0 & 0 & 0 & & \\
\vdots & \vdots & \vdots & & \\
0 & 0 & 0 & &
\end{array}\right),$$

where $\omega(R_2)=\omega(R_3)$ and $\omega(C_2)=\omega(C_3)$. It then follows by Claim 1 that $\omega(R_2)=\omega(R_3)\geqslant 5$ and $\omega(C_2)=\omega(C_3)\geqslant 5$. Clearly, B is an orthogonal matrix of order $n-3$ and type 1. If B were inseparable, then we would have

$$\omega(A)\geqslant\omega(B)+19\geqslant 5(n-3)-8+19>5n-8,$$

which is impossible. So B is separable.

Now suppose $\omega(R_2)=\omega(R_3)=5$ and $\omega(C_2)=\omega(C_3)=5$. Then A has the following format:

$$A = \left(\begin{array}{ccc|cc|ccc}
1 & 1 & 1 & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 \\
1 & 1 & 0 & 1 & 0\cdots 0 & 1 & 1 & 0\cdots 0 \\
1 & 0 & 1 & 1 & 0\cdots 0 & 1 & 1 & 0\cdots 0 \\
\hline
0 & 1 & 1 & & & & & \\
0 & 0 & 0 & & A_1 & & O & \\
\vdots & \vdots & \vdots & & & & & \\
0 & 0 & 0 & & & & & \\
\hline
0 & 1 & 1 & & & & & \\
0 & 1 & 1 & & & & & \\
0 & 0 & 0 & & O & & A_2 & \\
\vdots & \vdots & \vdots & & & & & \\
0 & 0 & 0 & & & & &
\end{array}\right).$$

This would imply that both the fourth column and the fourth row of A have weight 3, which is impossible by Claim 1. Therefore, we have $\omega(C_2)+\omega(R_2)\geqslant 12$.

CLAIM 4. *If A contains a column C of weight 3 such that each row in $\Omega(C)$ has weight at least 5, then we have $\tau(C) \geqslant 17$.*

Proof. Assume $\Omega(C) = \{R_1, R_2, R_3\}$. Then we have $\tau(C) = \omega(R_1) + \omega(R_2) + \omega(R_3) \geqslant 15$. Note that $\tau(C) \equiv 1 \pmod 4$ by Lemma 2.2. Therefore, we have $\tau(C) \geqslant 17$.

Now let $\mathfrak{R}$ (respectively, $\mathfrak{C}$) be the set of rows (respectively, columns) of weight 3 in A. A row R (respectively, a column C) in $\mathfrak{R}$ (respectively, $\mathfrak{C}$) is said to be *good* if there exists another row R^* (respectively, column C^*) in $\mathfrak{R}$ (respectively, $\mathfrak{C}$) such that $\omega(R, R^*) > 0$ (respectively, $\omega(C, C^*) > 0$). We then let $\mathfrak{R}_1$ (respectively, $\mathfrak{C}_1$) be the set of good rows (respectively, good columns) in $\mathfrak{R}$ (respectively, $\mathfrak{C}$); $\mathfrak{R}_2$ (respectively, $\mathfrak{C}_2$) be the set of rows (respectively, columns) in $\mathfrak{R} \backslash \mathfrak{R}_1$ (respectively, $\mathfrak{C} \backslash \mathfrak{C}_1$) such that each of the corresponding three columns (respectively, rows) has weight at least 5; $\mathfrak{R}_3$ be the set of rows in $\mathfrak{R} \backslash (\mathfrak{R}_1 \cup \mathfrak{R}_2)$ such that the total number of 1's in the corresponding three columns is at least 17; and $\mathfrak{C}_3$ be the set of columns in $\mathfrak{C} \backslash (\mathfrak{C}_1 \cup \mathfrak{C}_2)$ that does not cover any row in $\mathfrak{R}_3$. We also define $r_i = |\mathfrak{R}_i|$ and $c_i = |\mathfrak{C}_i|$ for $1 \leqslant i \leqslant 3$. Then by counting the number of 1's in each column of A, we have

$$\omega(A) \geqslant 5n - 2(c_1 + c_2 + c_3) + 2r_1 + 2r_2 + 2r_3. \tag{3}$$

Similarly, by counting the number of 1's in each row of A, we have

$$\omega(A) \geqslant 5n - 2(r_1 + r_2 + r_3) + 2c_1 + 2c_2 + 2c_3. \tag{4}$$

By adding (3) and (4) together, it follows that $\omega(A) \geqslant 5n$, which is a final contradiction. ∎

4.2. *Matrices of Type* 0

THEOREM 4.2. *We have*

$$f_0(m, n) = \begin{cases} 2(m+n) - 4 & \text{if } m \text{ and } n \text{ are both even} \\ \text{does not exist} & \text{if either } m \leqslant 5 \text{ or } n \leqslant 5; \text{ and } m \text{ is odd} \\ 4n & \text{if } m = 7 \text{ and } n \geqslant 6 \\ 2(m+n) - 2 & \text{if } m \geqslant 9 \text{ is odd and } n \geqslant 6 \text{ is even} \\ 2(m+n) + 4 & \text{if } m \text{ and } n \text{ are both odd}; m \leqslant n; \text{ and } m = 9 \text{ or } 11 \\ 2(m+n) & \text{if } m \geqslant 13; n \geqslant 13; \text{ and } m \text{ and } n \text{ are both odd.} \end{cases}$$

Proof. We first prove the upper bound. Recall that $A_{7,6}$ is the 7×6 matrix in Fig. 1a and A_7 is the 7×7 matrix in Fig. 1b. If m and n are both even, then we let A be the matrix in Fig. 2a. If $m \geqslant 9$ is odd and $n \geqslant 6$ is

$$\begin{pmatrix} J_{2,(n-2)} & J_2 \\ O & J_{(m-2),2} \end{pmatrix}$$

(a)

$$\left(\begin{array}{cc|cc} J_{2,(n-8)} & J_2 & J_2 & O \\ O & J_{(m-9),2} & O & O \\ \hline \multicolumn{2}{c|}{O} & \multicolumn{2}{c}{A_{7,6}} \end{array}\right)$$

(b)

FIGURE 2

even, then we let A be the matrix in Fig. 2b. If $m=7$ and $n \geqslant 6$ is even, then we let A be the matrix in Fig. 3a. If $m=7$ and $n \geqslant 7$ is odd, then we let A be the matrix in Fig. 3b.

We now assume m and n are both odd and $9 \leqslant m \leqslant n$. It can be easily seen that the following matrix A_9 is an inseparable orthogonal matrix of type 0 and order 9.

$$A_9 = \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{array}\right).$$

If $m=9$, then we let A be the matrix in Fig. 4a. If $m=11$, then we let A be the matrix in Fig. 4b. If $m \geqslant 13$, then we let A be the matrix in Fig. 5. Then we can easily verify that the matrix A constructed above is an $m \times n$ inseparable orthogonal matrix of type 0 that achieves the bound given in Theorem 4.2.

We now prove the lower bound. Lemma 1.3 implies that $f_0(m, n) \geqslant 2(m+n)-4$. In particular, we have $f_0(m, n) \geqslant 2(m+n)-2$ if m is odd and n is even by Lemma 2.1. Lemma 2.4 implies that $f_0(m, n) \geqslant 4n$ if $m=7$ and $n \geqslant 6$. We now assume and $m \geqslant 9$ and $n \geqslant 9$ are both odd. Suppose

$$\left(\begin{array}{c|c} A_{7,6} & \begin{matrix} J_{4,(n-6)} \\ O \end{matrix} \end{array}\right)$$

(a)

$$\left(\begin{array}{c|c} A_7 & \begin{matrix} J_{4,(n-7)} \\ O \end{matrix} \end{array}\right)$$

(b)

FIGURE 3

$$\left(\begin{array}{c|c} A_9 & \begin{matrix} J_{2,(n-9)} \\ O \end{matrix} \end{array}\right) \qquad \left(\begin{array}{c|c} A_9 & \begin{matrix} J_2 & O \\ O & O \end{matrix} \\ \hline O & J_{2,(n-9)} \end{array}\right)$$

(a) (b)

FIGURE 4

Theorem 4.2 is false. Then it follows by Lemma 2.1 that there exists an $m \times n$ inseparable matrix A of type 0 such that (i) $\omega(A) \leqslant 2(m+n)$; (ii) $\omega(A) < 2(m+n)$ if $m \geqslant 13$ and $n \geqslant 13$; and (iii) $m+n$ is minimal among such matrices.

Claim 1. *A contains a line of weight 2. In particular, if $m<n$, then A contains a column of weight 2.*

Proof. Suppose not. Then we have $\omega(\mathrm{A}) \geqslant 4n > 2(m+n)$ if $n>m$, which contradicts the choice of A. So we assume $n=m$. Then at least one line of A has weight larger than 4 by Lemma 2.5, which implies that $\omega(A) > 4n = 2(m+n)$, which is impossible.

Claim 2. *A contains no two identical lines of weight 2.*

Proof. Suppose not, and let B be a submatrix of A by deleting two identical columns of weight 2. Then B is an $m \times (n-2)$ inseparable matrix of type 0. Since $m+n$ is minimal, it follows that $n-2=7$, which implies that $m \geqslant n=9$. It then follows by Lemma 2.4 that

$$\omega(A) = \omega(B) + 4 \geqslant 4m + 4 > 2(m+n).$$

This proves Claim 2.

We now consider 2 cases.

Case 1. $n \geqslant m$ and $m=9$ or 11.

In this case, we can assume that A contains a column C_1 of weight 2 by Claim 1. Note that $\omega(A) \leqslant 2(m+n) \leqslant 4n$. Now let B be the submatrix

$$\left(\begin{array}{cc|cc} J_{2,(n-13)} & J_2 & \multicolumn{2}{c}{\begin{matrix} J_2 & O \\ O & O \end{matrix}} \\ O & J_{(m-13),2} & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{O}} & \begin{matrix} J_2 & O \\ O & O \end{matrix} & A_{7,6}^T \\ \cline{3-4} & & A_{7,6} & O \end{array}\right)$$

FIGURE 5

obtained from A by deleting column C_1 and the corresponding two rows. If $m=9$, then B is an admissible $7\times(n-1)$ orthogonal matrix of type 0 by Claim 2. It then follows by Lemma 2.4 that

$$\omega(A)\geqslant\omega(B)+8\geqslant 4(n-1)+8=4n+4,$$

which is impossible. So we assume $m=11$.

If A contains 2 columns of weight 2, then A has the following format:

$$A=\left(\begin{array}{cc|c|c} 1 & 0 & & \\ 1 & 0 & J_{4,t} & B_1 \\ 0 & 1 & & \\ 0 & 1 & & \\ \hline \multicolumn{2}{c|}{O} & O & B_2 \end{array}\right),$$

where B_2 has no zero column. Therefore, B_2 is an admissible $7\times(n-t-2)$ orthogonal matrix of type 0. It can be easily seen that $\omega(B_1)\geqslant 6$. It then follows by Lemma 2.4 that

$$\omega(A)\geqslant\omega(B_1)+\omega(B_2)+4t+4\geqslant 6+4(n-t-2)+4t+4=4n+2,$$

which is a contradiction. So A contains exactly one column of weight 2.

If every column C (other than C_1) in A such that $\Omega(C_1)\subseteq\Omega(C)$ has weight at least 6, then we would have $\omega(A)\geqslant 4n-2+6=4n+4$, which is impossible. So there exists a column, say C_2, of weight 4 in A such that $\Omega(C_1)\subseteq\Omega(C_2)$, and A has the following format:

$$A=\left(\begin{array}{cc|c|c} 1 & 1 & & \\ 1 & 1 & J_{4,t} & B_1 \\ 0 & 1 & & \\ 0 & 1 & & \\ \hline \multicolumn{2}{c|}{O} & O & B_2 \end{array}\right),$$

where B_2 has no zero column. Therefore, B_2 is an admissible $7\times(n-t-2)$ orthogonal matrix of type 0. It can be easily seen that $\omega(B_1)\geqslant 4$. It then follows by Lemma 2.4 that

$$\omega(A)\geqslant\omega(B_1)+\omega(B_2)+4t+6\geqslant 4+4(n-t-2)+4t+6=4n+2,$$

which is again a contradiction.

Case 2. $m\geqslant 13$, and $n\geqslant 13$.

We first assume that A contains a column C_1 of weight 2 such that $\tau(C_1)=8$. Let B be the submatrix obtained from A by deleting column C_1

and the corresponding two rows in $\Omega(C_1)$. Then B is an inseparable $(m-2)\times(n-1)$ orthogonal matrix of type 0. Therefore, we have

$$\omega(A)=\omega(B)+8\geqslant 2(m-2+n-1)-2+8=2(m+n),$$

which is impossible. Therefore, we have shown that for any column C of weight 2 in A, $\tau(C)\geqslant 12$. Similarly, for any row of weight 2 in A, the total number of 1's in the corresponding 2 columns is at least 12.

Let r (respectively, c) be the number of rows (resp., columns) of weight 2 in A. Then by counting the total number of 1's in the columns of A, we have

$$\omega(A)\geqslant 12r+2c+4(n-2r-c)=4n+4r-2c.$$

Similarly, we have $\omega(A)\geqslant 4m+4c-2r$. This implies that

$$\omega(A)\geqslant 2(m+n)+r+c\geqslant 2(m+n),$$

which is a final contradiction. ∎

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